UDC 539.3

# A VERSION OF THE COUPLE STRESS THEORY OF ELASTICITY FOR A ONE-DIMENSIONAL CONTINUOUS MEDIUM WITH INHOMOGENEOUS PERIODIC STRUCTURE

PMM Vol. 36, №6, 1972, pp. 1086-1093

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(Received August 3, 1972)

We consider a one-dimensional continuous medium composed of a number of periodically recurring homogeneous regions. In each of these regions the motion is defined by the dynamic equations of elasticity, the initial conditions and the conditions of continuity at the boundaries of the regions. We prove that the displacements of the region boundaries are determined by certain continuous functions. The finite difference equations defining these functions in the long wave approximation describe the frequency spectrum of the inhomogeneous medium with an accuracy, directly related to the number of the field functions introduced for the displacements of boundaries of the same type. In this manner we present a version of the couple stress theory of elasticity, in which the mean value of the displacement of the region boundaries can be treated as a displacement of a certain homogeneous macroscopic body, while the relative displacements of the boundaries (which may be called moments or couple stresses) define the interaction between the regions and, in particular, the stress concentration due to the inhomogeneity of the structure.

**1.** Let us consider the displacement $u\ (x, t)$ of an elastic rod of length $L$, consisting of $N$ different homogeneous segments. The initial and boundary conditions have the form

$$u\ (x,\ 0) = u^{\cdot}\ (x,\ 0) = 0$$

$$u\ (0,\ t) = U_1\ (t), \qquad u\ (L,\ t) = U_{N+1}\ (t)$$

We assume for convenience that

$$U_1(t) = U_{N+1}(t) = 0 \quad \text{for} \quad t < 0$$

The displacement $U_j\ (\xi_j,\ t)$ of each $j$th segment is determined from the wave equation, the initial conditions and the conditions of continuity at the regions boundaries

$$\frac{\partial^2 u_j}{\partial t^2} = \frac{\partial^2 u_j}{\partial \eta_j^2}, \qquad j = 1, 2 \ldots N$$

$$u_j(\eta_j, 0) = u_j^{\cdot}(\eta_j, 0) = 0$$

$$u_j(0, t) = U_j(t), \qquad u_j(\lambda_j, t) = U_{j+1}(t)$$

$$U_j(t) = 0 \quad \text{for} \quad t < 0 \tag{1.1}$$

$$s_j = \varkappa_j \left(\frac{\partial u_j}{\partial \eta_j}\right)_{\eta_j = 0} = \varkappa_{j-1} \left(\frac{\partial u_{j-1}}{\partial \eta_{j-1}}\right)_{\eta_{j-1} = \lambda_{j-1}}$$

$$s_{j+1}=\varkappa_j\left(\frac{\partial u_j}{\partial \eta_j}\right)_{\eta_j=\lambda_j}=\varkappa_{j+1}\left(\frac{\partial u_{j+1}}{\partial \eta_{j+1}}\right)_{\eta_{j+1}=0}$$

$$\eta_j=\frac{\xi_j}{c_j},\qquad \lambda_j=\frac{l_j}{c_j},\qquad \varkappa_j=\frac{E_jF_j}{c_j}$$

Here $c_j$ is the speed of sound for the $j$ th segment, $l_j$ is the length of the $j$ th segment, $\xi_j$ is the position (internal) coordinate on the $x$-axis measured from the left boundary of the $j$ th segment, $s_j$ and $s_{j+1}$ denote the respective stresses at the left and right boundary of the $j$ th segment, $E_j$ is the Young's modulus and $F_j$ is the cross section of the $j$th boundary.

Solving the boundary value problems with the conditions of continuity given by (1.1) by means of the Laplace-Carson transformation [1]

$$u^*(p)=p\int e^{-pt}u(t)\,dt$$

we obtain the following recurrence relations for the Laplace transforms of the boundary displacements $U_j^*(p)$:

$$\frac{\varkappa_j}{\operatorname{sh}\lambda_j p}U_{j+1}^*(p)-(\varkappa_j\operatorname{cth}\lambda_j p+\varkappa_{j-1}\operatorname{cth}\lambda_{j-1}p)\,U_j^*(p)+\frac{\varkappa_{j-1}}{\operatorname{sh}\lambda_{j-1}p}U_{j-1}^*(p)=0$$

$$j=2,3\ldots N-1 \tag{1.2}$$

It can be confirmed by direct substitution that (1.2) represent Laplace transforms of the following equations:

$$-\frac{\varkappa_j+\varkappa_{j-1}}{2}U_j(t)+\varkappa_j\sum_{n=1,3,5\ldots}U_{j+1}(t-n\lambda_j)+\varkappa_{j-1}\sum_{n=1,3,5\ldots}U_{j-1}(t-n\lambda_{j-1})=$$

$$\varkappa_j\sum_{m=2,4,6\ldots}U_j(t-m\lambda_j)+\varkappa_{j-1}\sum_{m=2,4,6\ldots}U_j(t-m\lambda_{j-1}) \tag{1.3}$$

$$j=2,3\ldots N-1$$

We eliminate the sums in (1.3) in the following manner. We write the equations for the time $t^*=t+2\lambda_j$ and subtract the corresponding equations given by (1.3) for the time $t$. We use the resulting equations and repeat the procedure to obtain the difference between the times $t$ and $t^{**}=t+2\lambda_{j-1}$. As a result we find

$$\frac{\varkappa_j+\varkappa_{j-1}}{2}\Delta_{\mu_j}U_j-\frac{\varkappa_j-\varkappa_{j-1}}{2}\Delta_{\nu_j}U_j=\varkappa_j\Delta_{\lambda_{j-1}}U_{j+1}+\varkappa_{j-1}\Delta_{\lambda_j}U_{j-1} \tag{1.4}$$

where

$$\Delta_\alpha U_j(t)=U_j(t+\alpha)-U_j(t-\alpha)$$

Thus the dynamic behavior of an inhomogeneous rod is described using a system of time difference equations, from which the displacements of all internal boundaries can be obtained when the displacements of the rod ends are given.

**2.** In an earlier paper [2] it was shown that for the one-dimensional continuous medium constructed on the basis of a linear inhomogeneous chain, if the number of the field functions introduced for the displacements of particles of the same type is increased (or, in other words, when the number of identical particles or cells included in the macrocell is increased), then the long wave approximation to the spectrum of the continuous medium agrees, with sufficient accuracy, with the spectrum of the initial inhomogeneous chain. Above we have shown that an analogous result can be obtained for the field functions corresponding to the displacements of the region boundaries of the same type, in an inhomogeneous rod possessing a periodic structure. With this in mind, let us first consider a two-component rod consisting of two types of homogeneous segments recurring periodically. In this case the system (1.4) has the form

$$\frac{1+\gamma}{2}\Delta_\mu U_j + \frac{1-\gamma}{2}\Delta_\nu U_j = \gamma\Delta_{\lambda_2}U_{j+1} + \Delta_{\lambda_1}U_{j-1}, \qquad j=3,5,7\ldots \tag{2.1}$$
$$\frac{1+\gamma}{2}\Delta_\mu U_j + \frac{1-\gamma}{2}\Delta_\nu U_j = \gamma\Delta_{\lambda_2}U_{j-1} + \Delta_{\lambda_1}U_{j+1}, \qquad j=2,4,6\ldots$$

where

$$\gamma = \frac{\varkappa_1}{\varkappa_2}, \qquad \mu = \lambda_1 + \lambda_2, \qquad \nu = \lambda_1 - \lambda_2$$

The spectral equation for such a rod is obtained by substituting into the system (2.1)

$$U_{2j} = U_1 \exp i\{kj(l_1+l_2) - \omega t\}$$
$$U_{2j+1} = U_2 \exp i\{k[j(l_1+l_2) - l_2] - \omega t\} \tag{2.2}$$
$$j = 1,2,3\ldots$$
$$\cos k(l_1+l_2) = \cos(\lambda_1+\lambda_2)\omega - \frac{(1-\gamma)^2}{2\gamma}\sin\lambda_1\omega\sin\lambda_2\omega \tag{2.3}$$

In the simplest case when the wave numbers $k$ and frequencies $\omega$ are small, this reduces to

$$k^2(l_1+l_2)^2 = \omega^2\left[\lambda_1^2 + \lambda_2^2 + \frac{(1+\gamma)^2}{2\gamma}\lambda_1\lambda_2\right]$$

The latter expression shows that for small $k$ and $\omega$ the speed of sound $c$ in a two-component rod has the form

$$c = (l_1+l_2)^2\left[\lambda_1^2 + \lambda_2^2 + \frac{(1+\gamma)^2}{2\gamma}\lambda_1\lambda_2\right]^{-1}$$

Expression (2.3) implies that real wave numbers $k$ do not exist for all values of $\omega$. The admissible frequencies must satisfy the following inequality

$$\left(\frac{1-\gamma}{1+\gamma}\right)^2\cos(\lambda_1-\lambda_2)\omega - \frac{4\gamma}{(1+\gamma)^2} \leqslant \cos(\lambda_1+\lambda_2)\omega \leqslant$$
$$\left(\frac{1-\gamma}{1+\gamma}\right)^2\cos(\lambda_1-\lambda_2)\omega + \frac{4\gamma}{(1+\gamma)^2} \tag{2.4}$$

The frequency ranges for the interval $3-2\sqrt{2} \leqslant \gamma \leqslant 3+2\sqrt{2}$ satisfying the condition (2.4) are shown in Fig. 1 as segments of the solid thick lines. Here we have

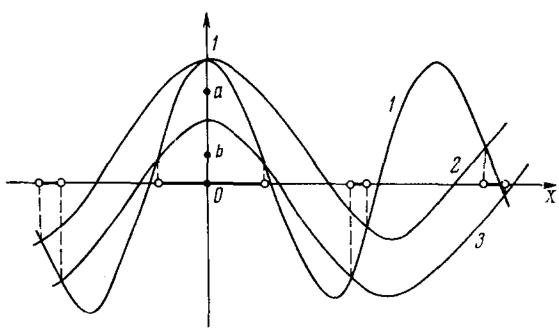


Fig. 1

$$X=(\lambda_1+\lambda_2)\,\omega,\quad a=\left(\frac{1-\gamma}{1+\gamma}\right)^2,\quad b=\frac{4\gamma}{(1+\gamma)^2},\quad \alpha=\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}$$

with $\cos X$ corresponding to curve *1* , $a\cos\alpha X+b$ to curve *2* and $a\cos\alpha X-b$ to curve *3*. For the remaining values of $\gamma$ the difference from the given diagram consists in the fact that $b>a$.

Thus the two-component rod with periodic structure considered here represents a distinctive wave filter which passes the waves with frequencies belonging to the infinite set of admissible frequency ranges (2.4). An equation analogous to (2.3) defines in quantum mechanics the energy levels in a particle in a periodic potential field, and this simulates the behavior of an electron in a one-dimensional crystal [3, 4].

Let us now introduce the continuous differentiable field functions $u_1(x,t)$ and $u_2(x,t)$ connected with the functions $U_j(t)$ by means of the following relations:

$$u_2(x,t)=U_j(t),\quad j=3,5,7\ldots\quad \text{for}\quad x=\left(\frac{j}{2}-\frac{1}{2}\right)(l_1+l_2)\tag{2.5}$$
$$u_1(x,t)=U_j(t),\quad j=2,4,6\ldots\quad \text{for}\quad x=\left(\frac{j}{2}-1\right)(l_1+l_2)+l_1$$

and satisfying, for any values of $x$, the equations

$$\frac{1+\gamma}{2}\Delta_\mu u_2(x)+\frac{1-\gamma}{2}\Delta_\nu u_2(x)=\gamma\Delta_{\lambda_2}u_1(x+l_1)+\Delta_{\lambda_1}u_1(x-l_2)$$
$$\frac{1+\gamma}{2}\Delta_\mu u_1(x)+\frac{1-\gamma}{2}\Delta_\nu u_1(x)=\gamma\Delta_{\lambda_2}u_2(x-l_1)+\Delta_{\lambda_1}u_2(x+l_2)\tag{2.6}$$

The latter equations reduce to (2.1) when $x$ are taken from (2.5). We obtain the long wave approximation to (2.1) from (2.6) by expanding, in the latter, $u_1(x+l_1,t)$, $u_1(x-l_2,t)$, $u_2(x+l_2,t)$ and $u_2(x-l_1,t)$ in powers of $l$ up to and including the $l^2$-terms

$$\frac{1+\gamma}{2}\Delta_\mu u_2+\frac{1-\gamma}{2}\Delta_\nu u_2=\gamma\Delta_{\lambda_2}u_1+\gamma l_1\Delta_{\lambda_2}u_1'+\gamma\frac{l_1^2}{2}\Delta_{\lambda_2}u_1''+$$
$$\Delta_{\lambda_1}u_1-l_2\Delta_{\lambda_2}u_1'+\frac{l_2^2}{2}\Delta_{\lambda_1}u_1''$$
$$\frac{1+\gamma}{2}\Delta_\mu u_1+\frac{1-\gamma}{2}\Delta_\nu u_1=\gamma\Delta_{\lambda_2}u_2-\gamma l_1\Delta_{\lambda_2}u_2'+\frac{\gamma l_1^2}{2}\Delta_{\lambda_2}u_2''+$$
$$\Delta_{\lambda_1}u_2+l_2\Delta_{\lambda_1}u_2'+\frac{l_2^2}{2}\Delta_{\lambda_1}u_2''\tag{2.7}$$

Here the prime denotes a derivative in $x$. The spectral equation corresponding to (2.7) has the form

$$1-\frac{1}{2}k^2(l_1+l_2)^2=\cos(\lambda_1+\lambda_2)\,\omega-\frac{(1-\gamma)^2}{2\gamma}\sin\lambda_1\omega\sin\lambda_2\omega\tag{2.8}$$

and represents an expansion of the exact spectral equation (2.3) in $k$ about the point $k=0$. The admissible frequencies of (2.8) are obtained from the condition that $k^2>0$, and must satisfy the following inequality:

$$\cos(\lambda_1+\lambda_2)\,\omega\leqslant\frac{4\gamma}{(1+\gamma)^2}+\left(\frac{1-\gamma}{1+\gamma}\right)^2\cos(\lambda_1+\lambda_2)\,\omega$$

Clearly, the admissible frequencies which correspond to other values of $k$, e.g.

$$k = \frac{\pi}{l_1 + l_2}, \qquad k = \frac{\pi}{2(l_1 + l_2)}, \qquad k = \frac{3\pi}{2(l_1 + l_2)}$$

do not appear in the present case involving two field functions.

The proposed method which uses field functions in the long wave approximation to provide a more accurate description of the dynamics of an inhomogeneous medium, is based on the fact that additional, different field functions are introduced for the boundaries of the same type of segments and particles. In other words, a macrocell is constructed, the period of which is an integral multiple of the minimal period of the structure and exceeds it in value. Each particle boundary in the macrocell is given its own field function.

In accordance with the above statements, let us consider a two-component rod with a macrocell containing four consecutive segments, i.e. we introduce four field functions $u_1(x, t)$, $u_2(x, t)$, $u_3(x, t)$ and $u_4(x, t)$ such, that

$$\begin{aligned}
&u_1(x,t) = U_j(t), \quad j = 2, 6, 10\ldots \quad \text{for } x = \left(\frac{j}{2} - 1\right)(l_1 + l_2) + l_1 \\
&u_2(x,t) = U_j(t), \quad j = 3, 7, 11\ldots \quad \text{for } x = \left(\frac{j}{2} - \frac{1}{2}\right)(l_1 + l_2) \\
&u_3(x,t) = U_j(t), \quad j = 4, 8, 12\ldots \quad \text{for } x = \left(\frac{j}{2} - 1\right)(l_1 + l_2) + l_1 \\
&u_4(x,t) = U_j(t), \quad j = 5, 9, 13\ldots \quad \text{for } x = \left(\frac{j}{2} - \frac{1}{2}\right)(l_1 + l_2)
\end{aligned} \tag{2.9}$$

In the long wave approximation these functions satisfy the following equations:

$$\begin{aligned}
\frac{1+\gamma}{2}\Delta_\mu u_1 + \frac{1-\gamma}{2}\Delta_\nu u_1 &= \gamma\Delta_{\lambda_2}u_4 - \gamma l_1\Delta_{\lambda_2}u_4' + \frac{\gamma l_1^2}{2}\Delta_{\lambda_2}u_4'' + \\
&\quad \Delta_{\lambda_1}u_2 + l_2\Delta_{\lambda_1}u_2' + \frac{l_2^2}{2}\Delta_{\lambda_1}u_2'' \\
\frac{1+\gamma}{2}\Delta_\mu u_2 + \frac{1-\gamma}{2}\Delta_\nu u_2 &= \gamma\Delta_{\lambda_2}u_3 + \gamma l_1\Delta_{\lambda_2}u_3' + \frac{\gamma l_1^2}{2}\Delta_{\lambda_2}u_3'' + \\
&\quad \Delta_{\lambda_1}u_1 - l_2\Delta_{\lambda_1}u_1' + \frac{l_2^2}{2}\Delta_{\lambda_1}u_1'' \\
\frac{1+\gamma}{2}\Delta_\mu u_3 + \frac{1-\gamma}{2}\Delta_\nu u_3 &= \gamma\Delta_{\lambda_2}u_2 - \gamma l_1\Delta_{\lambda_2}u_2' + \frac{\gamma l_1^2}{2}\Delta_{\lambda_2}u_2'' + \\
&\quad \Delta_{\lambda_1}u_4 + l_2\Delta_{\lambda_1}u_4' + \frac{l_2^2}{2}\Delta_{\lambda_1}u_4'' \\
\frac{1+\gamma}{2}\Delta_\mu u_4 + \frac{1-\gamma}{2}\Delta_\nu u_4 &= \gamma\Delta_{\lambda_2}u_1 + \gamma l_1\Delta_{\lambda_2}u_1' + \frac{\gamma l_1^2}{2}\Delta_{\lambda_2}u_1'' + \\
&\quad \Delta_{\lambda_1}u_3 - l_2\Delta_{\lambda_1}u_3' + \frac{l_2^2}{2}\Delta_{\lambda_1}u_3''
\end{aligned} \tag{2.10}$$

In this case the spectral equation decomposes into two equations

$$\pm\left[1 - \frac{1}{2}k^2(l_1 + l_2)^2\right] = \cos(\lambda_1 + \lambda_2)\omega - \frac{(1-\gamma)^2}{2\gamma}\sin\lambda_1\omega\sin\lambda_2\omega \tag{2.11}$$

One of these equations is identical to (2.8) and the other coincides with the expansion, in terms of $k$, of the exact spectral equation (2.3) about the point $k = \pi / (l_1 + l_2)$. The admissible frequencies defined by (2.11) are obtained from the condition that $k^2 > 0$, and satisfy the following relations:

$$\left(\frac{1-\gamma}{1+\gamma}\right)^2 \cos(\lambda_1-\lambda_2)\,\omega - \frac{4\gamma}{(1+\gamma)^3} \leqslant \cos(\lambda_1+\lambda_2)\,\omega \leqslant$$
$$\left(\frac{1-\gamma}{1+\gamma}\right)^2 \cos(\lambda_1-\lambda_2)\,\omega + \frac{4\gamma}{(1+\gamma)^2}$$

Further sharpening of the spectrum of the two-component rod is achieved by introducing eight field functions $u_m(x,t)$, $m=1,2,\ldots,8$. Their equations in the long wave approximation are

$$\frac{1+\gamma}{2}\Delta_\mu u_j + \frac{1-\gamma}{2}\Delta_\nu u_j = \gamma\Delta_{\lambda_2}u_{j-1} - \gamma l_1\Delta_{\lambda_2}u'_{j-1} + \frac{\gamma l_1^2}{2}\Delta_{\lambda_2}u''_{j-1} +$$
$$\Delta_{\lambda_1}u_{j+1} + l_2\Delta_{\lambda_1}u'_{j+1} + \frac{l_2^2}{2}\Delta_{\lambda_1}u''_{j+1}, \quad j=3,5,7$$
$$\frac{1+\gamma}{2}\Delta_\mu u_j + \frac{1-\gamma}{2}\Delta_\nu u_j = \gamma\Delta_{\lambda_2}u_{j+1} + \gamma l_1\Delta_{\lambda_2}u'_{j+1} + \frac{\gamma l_1^2}{2}\Delta_{\lambda_2}u''_{j+1} +$$
$$\Delta_{\lambda_1}u_{j-1} - l_2\Delta_{\lambda_1}u'_{j-1} + \frac{l_2^2}{2}\Delta_{\lambda_1}u''_{j-1}, \quad j=2,4,6,8 \tag{2.12}$$

The function $u_{j-1}(x,t)$ in the above system must be replaced by $u_8(x,t)$ for $j=1$ and $u_{j+1}$, $(x,t)$ by $u_1(x,t)$ for $j=8$, at the corresponding points. The spectral equation of (2.12) decomposes into the following four equations:

$$\pm\left[1-\frac{1}{2}k^2(l_1+l_2)^2\right] = \cos(\lambda_1+\lambda_2)\,\omega - \frac{(1-\gamma)^2}{2\gamma}\sin\lambda_1\omega\sin\lambda_2\omega \tag{2.13}$$
$$\pm k(l_1+l_2) = \cos(\lambda_1+\lambda_2)\,\omega - \frac{(1-\gamma)^2}{2\gamma}\sin\lambda_1\omega\sin\lambda_2\omega$$

The first two of these equations are identical with (2.8) and (2.11), while the remaining two represent expansions in terms of $k$ of the exact spectral equation (2.3) about the points

$$k=\frac{\pi}{2(l_1+l_2)}, \qquad k=\frac{3\pi}{2(l_1+l_2)}$$

respectively.

Thus, on increasing the number of the field functions introduced for the displacements of boundaries of the same type in a two-component system we find, that in the long wave approximation its spectral characteristics are sharpened and on doubling the number of the field functions the spectral equation is obtained about the points

$$k=0, \quad k=\frac{\pi}{l_1+l_2}, \quad k=\frac{3\pi}{2(l_1+l_2)}, \quad k=\frac{\pi}{2(l_1+l_2)}$$

as well as other points obtained by halving the preceding intervals.

The above example of doubling the number of particles included in the macrocell shows at once that the preceding frequency ranges are always included in the following ones, i.e. in the limit the correct spectrum is obtained accurately. It should be stated however, that any alternative method of inctreasing the number of segments in the macrocell will, in the limit, yield the same result.

The fact established above forms the basis of the proposed method of constructing the couple stress theory of elasticity, the latter employing a finite number of field functions to increase the accuracy of the macroscopic representation in the homogeneous-in-the-mean and isotropic elastic systems and in the polycrystalline solids.

The field equations for the two-particle (2.7), four-particle (2.10) and eight-particle (2.12) macrocells, applicable to a two-component inhomogeneous rod, are of the following type. The dependence on the coordinates with respect to displacements of the internal boundaries is given by a system of second order linear differential equations, while the corresponding time dependence is expressed in terms of a system of finite-difference equations. Further simplification of these systems by expanding them with respect to time and retaining the cubic terms is expedient only when the loads vary, or the boundaries of the region (the rod ends) are displaced, at a sufficiently slow rate. In general, this leads to loss of certain important features of the frequency spectrum of the system.

Solutions of the boundary value problems for a finite rod based on (2.7), (2.10) and (2.12) can be obtained using either the ordinary Fourier transform

$$u_j(x, t) = X_j(x)\, T_j(t)$$

or the Laplace-Carson transform, as well as other integral transforms, and in all cases the corresponding spectral equations (2.8), (2.11) and (2.13) play a vital role.

**3.** The most general case of a one-dimensional, piecewise homogeneous periodic elastic medium represents a system of periodically recurring groups of segments (layers). Let us write the long wave approximation to the field equations for the displacements of the segment boundaries within the group, obtained from (1.6) in the same manner as in Sect. 2. We denote by $u_j\ (x, t)$ the field displacement corresponding to the real displacement of the left boundary of the $j$th segment within a group containing $n$ segments

$$\frac{\varkappa_j + \varkappa_{j-1}}{2} \Delta_{\mu_j} u_j - \frac{\varkappa_j - \varkappa_{j-1}}{2} \Delta_{\nu_j} u_j = \varkappa_j \Delta_{\lambda_{j-1}} u_{j+1} + \varkappa_j l_j \Delta_{\lambda_{j-1}} u'_{j+1} +$$
$$\frac{\varkappa_j l_j^2}{2} \Delta_{\lambda_{j-1}} u''_{j+1} + \varkappa_{j-1} \Delta_{\lambda_j} u_{j-1} - \varkappa_{j-1} l_{j-1} \Delta_{\lambda_j} u'_{j-1} + \frac{\varkappa_{j-1} l_{j-1}^2}{2} \Delta_{\lambda_j} u''_{j-1}. \qquad (3.1)$$

For $j = 1$ the function $u_{j-1}\ (x, t)$ must be replaced by $u_n\ (x, t)$, and for $j = n$, $u_{j+1}\ (x, t)$ by $u_1\ (x, t)$, at the corresponding points.

In the version of the couple stress theory of elasticity considered here the system (3.1) is the most general one, since it yields equations covering all specific examples of one-dimensional periodic structures. Indeed, let the minimal group of periodically recurring segments consist of $m$ segments and let introduce $q$ field functions for each, materially different segments. By setting $n = qm$ we obtain, from (3.1), a system of field equations for a macrocell.

As an example we consider an inhomogeneous medium consisting of four, periodically recurring, different layers. The spectral equation of this medium is

$$2 \cos k\,(l_1 + l_2 + l_3 + l_4) = 2 \cos \lambda_1\omega \cos \lambda_2\omega \cos \lambda_3\omega \cos \lambda_4\omega + (\gamma_1\gamma_3 + \gamma_2\gamma_4) \times$$
$$\sin \lambda_1\omega \sin \lambda_2\omega \sin \lambda_3\omega \sin \lambda_4\omega - (\gamma_1\gamma_2 + \gamma_3\gamma_4) \sin \lambda_2\omega \sin \lambda_4\omega \cos \lambda_1\omega \cos \lambda_3\omega -$$
$$(\gamma_1\gamma_4 + \gamma_2\gamma_3) \sin \lambda_1\omega \sin \lambda_3\omega \cos \lambda_2\omega \cos \lambda\ \omega - \sum_{i=1}^{4} \frac{\gamma_i^2 + 1}{\gamma_i} \sin \lambda_i\omega \sin \lambda_{i-1}\omega \times$$
$$\cos \lambda_{i+1}\omega \cos \lambda_{i+2}\omega$$
$$\gamma_1 = \frac{\varkappa_1}{\varkappa_4}, \quad \gamma_2 = \frac{\varkappa_2}{\varkappa_1}, \quad \gamma_3 = \frac{\varkappa_3}{\varkappa_2}, \quad \gamma_4 = \frac{\varkappa_4}{\varkappa_3} \qquad (3.2)$$

It is clear that the above equation contains, in the long wave approximation, the spectral equations of a two-component medium for both two-particle and four-particle macrocells, as particular cases.

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Translated by L.K.

UDC 539.385

# TORSION OF AN AXISYMMETRIC ANISOTROPIC BODY WITH MIXED BOUNDARY CONDITIONS ON THE SIDE SURFACE

PMM Vol. 36, №6, 1972, pp.1094-1099
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(Received March 6, 1972)

Differential and integral operators are used to solve the nonsymmetric system of equations characterizing the pure torsion of a body of revolution with variable shear moduli. The stress and displacement functions are expressed by convergent series containing two arbitrary analytic functions of a complex variable and the coefficients of a real argument defined in terms of the shear modulus. As an illustration, the problem of torsion of a hollow cylinder with mixed boundary conditions is considered. The torsion of isotropic rods has been examined in detail in [1], and for anisotropic bodies of revolution in [2, 3].

**1. Initial equations.** The pure torsion of a body of revolution whose axis of cylindrical inhomogeneous anisotropy coincides with the geometric body axis is characterized in the cylindrical coordinates $r\ z\ \theta$ by a linear system of partial differential equations of elliptic type [2]

$$\frac{\partial \varphi}{\partial r} - P(r)\frac{\partial \psi}{\partial z} = 0, \qquad \frac{\partial \varphi}{\partial z} + Q(r)\frac{\partial \psi}{\partial r} = 0$$
$$P(r) = r^3 G_1(r), \qquad Q(r) = r^3 G_2(r) \tag{1.1}$$

Here $\varphi$ is the stress function, $\psi$ is the displacement function, $G_{z\theta} = G_1(r)$, $G_{r\theta} = G_2(r)$ are the shear moduli of the corresponding planes which we consider given (or found from experiment), bounded in a range of variation, and piecewise-continuous functions of the single variable $r$. Two stress components $\tau_{z\theta} = \tau_1(r, z)$, $\tau_{r\theta} = \tau_2(r, z)$ and the tangential displacement $u_\theta = v(r, z)$ defined by the formulas

$$\tau_1 = \frac{1}{r^2}\frac{\partial \varphi}{\partial r} = rG_1(r)\frac{\partial \psi}{\partial z}, \quad \tau_2 = -\frac{1}{r^2}\frac{\partial \varphi}{\partial z} = rG_2(r)\frac{\partial \psi}{\partial r}, \quad v = r\psi \tag{1.2}$$